4 v/cm. From $E = \pi f \mu H (R + r)$ we obtain by setting $r \ll R$ and $\mu H = 15{,}000$ gauss, $R = 140$ cm for $f = 60$ Hz, and $R = 47$ cm for $f = 180$ Hz. It is seen that in this case, where we can consider E as independent of r, the effect of frequency conversion is even more beneficial since R may be reduced in direct proportion to f.

The dissipated power is $P = 2\pi REI$. With $I = 500$ amp we obtain

$$P = 1760 \text{ kw for } f = 60 \text{ Hz}$$

and

$$P = 590 \text{ kw for } f = 180 \text{ Hz}$$

For operation with nitrogen or air where maintenance fields are several times higher than in argon[11] minimum core radii at 180 Hz can be expected to be in the meter range and power levels in the megawatt range unless one wants to operate below atmospheric pressure. For windtunnel heaters such power levels are not uncommon. It is only with air and other reactive gases that the advantages of the electrodeless heating become important and only at these power levels that the installation and operating costs of rf heaters become prohibitive.

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Dynamic Equilibrium of a Compound Pendulum in an Artificial Satellite

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The equation of motion of a rigid body which is constrained to rotate about an axis fixed in an artificial satellite is presented, and the stability of particular solutions of this equation is studied both for Earth-pointing and rotating satellites. Two illustrative examples indicate possible uses of such a system as a device for establishing an attitude reference or for detecting variations in satellite rotation rates.

1. Introduction

IN a paper[1] read before the Royal Irish Academy only about one year after the launching of the first artificial satellite of the Earth, J. L. Synge discussed the behavior of a pendulum attached to such a satellite, taking the pendulum to be a particle fastened to the mass center of the satellite by means of a light rod and a universal joint. In this analysis, the effect of the pendulum on the motion of the satellite was presumed to be negligible, and it is this presumption that distinguishes both Synge's paper and the present one from the many that have dealt in the intervening years with pendulum-like devices in orbit. The difference between our work and that of Synge is that we take the pendulum to be a rigid body, rather than a particle, and constrain this body to rotate about an axis fixed in the satellite in an arbitrary position.

The system under consideration is described in detail in Sec. 2. In Sec. 3, the equation of motion is presented, and the stability of particular solutions of this equation is then analyzed in Sec. 4, which also contains illustrative examples intended to point out possible practical applications. Finally, friction effects are discussed briefly in Sec. 5.

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2. System Description

The system to be analyzed is shown schematically in Fig. 1, where O designates a particle fixed in an inertial reference frame, B is an artificial satellite whose mass center, Q, moves in a circular orbit centered at O, and C is a rigid body (compound pendulum) constrained to rotate relative to B about an axis fixed in both B and C and parallel to a unit vector $\mathbf{b}_x$.

To delineate permissible attitude motions of B, we introduce an orbiting reference frame A in which mutually perpendicular unit vectors $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ are fixed as shown in Fig. 2, and require B to rotate in A about an axis passing through Q and parallel to $\mathbf{a}_3$. The angle ϕ between $\mathbf{a}_2$ and any unit vector $\mathbf{b}_2$ that is fixed in B and perpendicular to $\mathbf{a}_3$ then furnishes a complete description of the orientation of B relative to A.

A seemingly complicated, but ultimately particularly convenient, choice of $\mathbf{b}_2$ may be made as follows: Designating as P the mass center of the pendulum C (see Fig. 2), and as S the foot of the perpendicular dropped from P on the axis of rotation of C, draw a line through S and parallel to $\mathbf{a}_3$, calling T the point of intersection of this line with the orbit plane; then let $\mathbf{b}_2$ point from Q toward T.

The orientation of C relative to B is characterized by the angle θ between a unit vector $\mathbf{c}_1$, directed from S toward P, and a unit vector $\mathbf{b}_y$ defined as follows: If the axis of rotation is parallel to $\mathbf{a}_3$, make $\mathbf{b}_x = \mathbf{a}_3$ and $\mathbf{b}_y = \mathbf{b}_2$. Otherwise, let

$$\mathbf{b}_y = \mathbf{b}_x \times \mathbf{a}_3[(\mathbf{b}_x \times \mathbf{a}_3)^2]^{-1/2} \tag{1}$$

In addition to those already mentioned, it is useful to introduce unit vectors $\mathbf{b}_1$, $\mathbf{b}_3$, $\mathbf{c}_2$, and $\mathbf{c}_3$, all of which are shown in Fig. 3. The unit vectors of each triplet form a dextral, orthogonal set, and each unit vector is fixed in the body designated by the associated letter. Thus, for example, $\mathbf{b}_1$ is fixed in B, whereas $\mathbf{c}_2$ is fixed in C.

The following dimensions are of interest: r, the orbital radius; b, the distance from Q to T; c, the distance from T to S; and l, the length of the line joining P and S (see Fig. 2).

As for inertia properties, m denotes the mass of C, and I_{ij}, defined in terms of the inertia dyadic $\mathbf{I}$ of C for P as

$$I_{ij} = \mathbf{c}_i \cdot \mathbf{I} \cdot \mathbf{c}_j, \qquad (i,j = 1,2,3) \tag{2}$$

denotes a typical product of inertia. Finally, the symbols x_i and y_i are defined as follows:

$$x_i = \mathbf{b}_x \cdot \mathbf{b}_i, \qquad (i = 1,2,3) \tag{3}$$

and

$$y_j = \mathbf{b}_y \cdot \mathbf{b}_j, \qquad (j = 1,2) \tag{4}$$

3. Equation of Motion

A differential equation governing θ is obtained by forming the generalized force F and the kinetic energy K of C and then using the relationship

$$(d/dt\partial)(K/\partial\dot{\theta}) - \partial K/\partial\theta = F \tag{5}$$

If C is supported in B by frictionless bearings, a suitable expression for F may be constructed by neglecting gravitational forces exerted on C by B and replacing the system of gravitational forces exerted on C by O with a gravitational couple of torque $\mathbf{T}$ and a gravitational force $\mathbf{F}$, applied at P, and then setting[2]

$$F = \mathbf{v}_{\dot{\theta}} \cdot \mathbf{F} + \boldsymbol{\omega}_{\dot{\theta}} \cdot \mathbf{T} \tag{6}$$

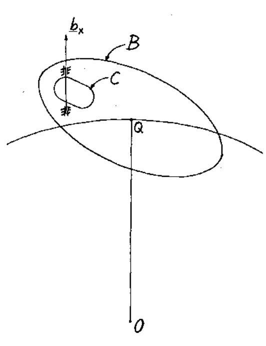

Fig. 1 Satellite and pendulum.

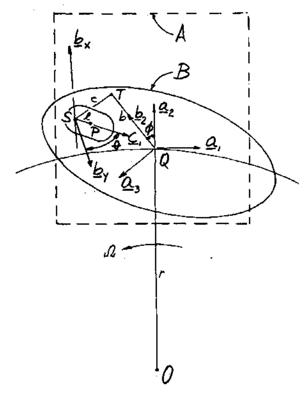

Fig. 2 Unit vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, $\mathbf{b}_2$, $\mathbf{b}_x$, $\mathbf{b}_y$ and $\mathbf{c}_1$.

where $\mathbf{v}_{\dot{\theta}}$ and $\boldsymbol{\omega}_{\dot{\theta}}$ are given by

$$\mathbf{v}_{\dot{\theta}} = l\mathbf{c}_2, \qquad \boldsymbol{\omega}_{\dot{\theta}} = \mathbf{c}_3 \tag{7}$$

and $\mathbf{F}$ and $\mathbf{T}$ are expressed with sufficient accuracy as[3]

$$\mathbf{F} = -GMm(\mathbf{p}^2)^{-3/2} \tag{8a}$$

$$\mathbf{T} = 3GM(\mathbf{p}^2)^{-5/2}\mathbf{p} \times \mathbf{I}\cdot\mathbf{p} \tag{8b}$$

Here G is the universal gravitational constant, M is the mass of O, and $\mathbf{p}$ is the position vector of P relative to O. Furthermore, letting Ω denote the orbital angular speed of point Q, one may set

$$GM = \Omega^2 r^3 \tag{9}$$

Assuming that b, c, and l are small in comparison with r, and confining attention to situations in which B rotates with constant (positive, negative, or zero) angular speed in A, so that ϕ can be expressed as

$$\phi = \phi_0 + \omega t \tag{10}$$

where ϕ_0 and ω are constants and t denotes time, one arrives by substitution from Eqs. (6–10) into Eq. (5) at the somewhat formidable appearing equation of motion

$$\begin{aligned}
&(ml^2 + I_{33})\ddot{\theta} + \\
&\{[mbly_2 + I_{23}x_3(x_1y_2 - x_2y_1)](\Omega + \omega)^2 + \\
&[mbl(-y_2 + 3y_1 \sin\phi \cos\phi + 3y_2 \cos^2\phi) + \\
&3I_{13}(x_1 \sin\phi + x_2 \cos\phi)(y_1 \sin\phi + y_2 \cos\phi) + \\
&3I_{23}x_3(x_1 \sin\phi + x_2 \cos\phi)(y_1 \cos\phi - y_2 \sin\phi)]\Omega^2\} \sin\theta + \\
&\{-x_3[mbly_1 + I_{13}(x_1y_2 - x_2y_1)](\Omega + \omega)^2 + \\
&[mblx_3(y_1 + 3y_2 \sin\phi \cos\phi - 3y_1 \cos^2\phi) + \\
&mcl(x_1y_2 - x_2y_1) + \\
&3I_{13}x_3(x_1 \sin\phi + x_2 \cos\phi)(-y_1 \cos\phi + y_2 \sin\phi) + \\
&3I_{23}(x_1 \sin\phi + x_2 \cos\phi)(y_1 \sin\phi + y_2 \cos\phi)]\Omega^2\} \cos\theta + \\
&\{\tfrac{1}{2}(ml^2 - I_{11} + I_{22})(x_1^2 + x_2^2)(\Omega + \omega)^2 + \\
&[-\tfrac{3}{2}(ml^2 - I_{11} + I_{22})\langle(x_3^2y_1^2 - y_2^2) \cos^2\phi + \\
&(x_3^2y_2^2 - y_1^2) \sin^2\phi - (1 + x_3^2)y_1y_2 \sin 2\phi\rangle + \\
&3I_{12}x_3\langle(y_1^2 - y_2^2) \sin 2\phi + 2y_1y_2 \cos 2\phi\rangle]\Omega^2\} \sin 2\theta + \\
&\{-I_{12}(x_1^2 + x_2^2)(\Omega + \omega)^2 + \\
&[-\tfrac{3}{2}(ml^2 - I_{11} + I_{22})x_3\langle(y_1^2 - y_2^2) \sin 2\phi + 2y_1y_2 \cos 2\phi\rangle + \\
&3I_{12}\langle(1 + x_3^2)y_1y_2 \sin 2\phi + (y_1^2 - x_3^2y_2^2) \sin^2\phi + \\
&(y_2^2 - x_3^2y_1^2) \cos^2\phi\rangle]\Omega^2\} \cos 2\theta = 0
\end{aligned} \tag{11}$$

However, it may be verified that this equation can be cast into the rather more attractive forms

$$\ddot{\theta} + f_1(t) \sin\theta + f_2(t) \cos\theta + f_3(t) \sin 2\theta + f_4(t) \cos 2\theta = 0 \tag{12}$$

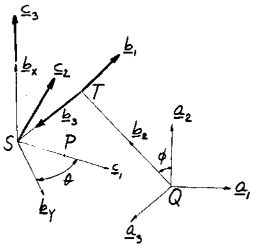

Fig. 3 Unit vectors $\mathbf{b}_1$, $\mathbf{b}_3$, $\mathbf{c}_2$, and $\mathbf{c}_3$.

and

$$\ddot{\theta} + F_1(\theta)\sin 2(\phi_0 + \omega t) + F_2(\theta)\cos 2(\phi_0 + \omega t) + F_3(\theta) = 0 \quad (13)$$

where $f_i(i = 1, \ldots, 4)$ is a known function of t, and $F_i(i = 1,2,3)$ is a known function of θ.

4. Stability of Dynamic Equilibrium States

When the pendulum C remains at rest in the satellite B, so that θ has a constant value, say θ^*, the pendulum is said to be in a state of dynamic equilibrium. To study the stability of such states, it is convenient to consider separately situations in which ω [see Eq. (10)] is equal to zero, hereafter treated under the heading "Earth-pointing satellites," and those in which ω differs from zero, discussed under the heading "Rotating satellites."

Earth-pointing Satellites

If $\omega = 0$, then it follows from Eq. (10) that ϕ has the constant value ϕ_0, and comparison of Eqs. (11) and (12) reveals that $f_1, \ldots, f_4$ are all constants. Consequently, Eq. (12) is satisfied for all t whenever θ has a constant value θ^* such that

$$f_1 \sin\theta^* + f_2 \cos\theta^* + f_3 \sin 2\theta^* + f_4 \cos 2\theta^* = 0 \quad (14)$$

Furthermore, if two new dependent variables, q_1 and q_2, are introduced by means of the definitions

$$q_1 = \theta - \theta^*, \qquad q_2 = \dot{\theta} \quad (15)$$

then Eq. (12) can be replaced with the two first-order equations

$$\dot{q}_1 = q_2 \quad (16)$$

$$\dot{q}_2 = -[f_1 \sin(\theta^* + q_1) + f_2 \cos(\theta^* + q_1) + f_3 \sin 2(\theta^* + q_1) + f_4 \cos 2(\theta^* + q_1)] \quad (17)$$

which, in view of Eq. (14), possess the solution $q_1 = q_2 = 0$, the stability or instability of which implies the stability or instability of the solution $\theta = \theta^*$ of Eq. (12).

Consider now the function $V(q_1,q_2)$ defined as

$$V(q_1,q_2) = f_1[\cos\theta^* - \cos(\theta^* + q_1)] - f_2[\sin\theta^* - \sin(\theta^* + q_1)] + \tfrac{1}{2}\{f_3[\cos 2\theta^* - \cos 2(\theta^* + q_1)] - f_4[\sin\theta^* - \sin 2(\theta^* + q_1)]\} + \tfrac{1}{2}q_2^2 \quad (18)$$

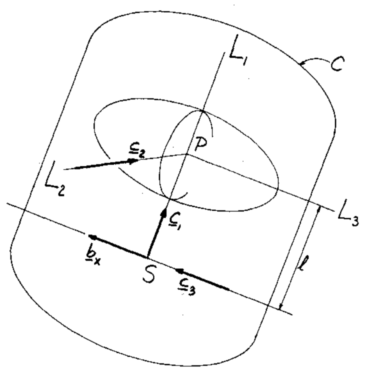

Fig. 4 Placement of axis of rotation in C.

This function vanishes when $q_1 = q_2 = 0$, and its total time-derivative vanishes for all t whenever q_1 and q_2 satisfy Eqs. (16) and (17). Hence the solution $q_1 = q_2 = 0$ of Eqs. (16) and (17) is stable (but not necessarily asymptotically stable) whenever V is positive definite; and, since $V(0,0) = 0$, V is positive definite whenever

$$\left.\frac{\partial^2 V}{\partial q_1^2}\right|_{q_1=q_2=0} > 0 \quad (19)$$

and

$$\left.\begin{vmatrix} \dfrac{\partial^2 V}{\partial q_1^2} & \dfrac{\partial^2 V}{\partial q_1 \partial q_2} \\ \dfrac{\partial^2 V}{\partial q_1 \partial q_2} & \dfrac{\partial^2 V}{\partial q_2^2} \end{vmatrix}\right|_{q_1=q_2=0} > 0 \quad (20)$$

Now, from Eq. (18)

$$\left.\frac{\partial^2 V}{\partial q_1^2}\right|_{q_1=q_2=0} = f_1 \cos\theta^* - f_2 \sin\theta^* + 2f_3 \cos 2\theta^* - 2f_4 \sin 2\theta^* \quad (21)$$

$$\left.\frac{\partial^2 V}{\partial q_2^2}\right|_{q_1=q_2=0} = 1 \quad (22)$$

and

$$\left.\frac{\partial^2 V}{\partial q_1 \partial q_2}\right|_{q_1=q_2=0} = 0 \quad (23)$$

Consequently, $\theta = \theta^*$ is a *stable* solution of Eq. (12) whenever θ^* satisfies Eq. (14) and

$${}_1 \cos\theta^* - f_2 \sin\theta^* + 2f_3 \cos 2\theta^* - 2f_4 \sin 2\theta^* > 0 \quad (24)$$

The fact that reversal of the inequality in this relationship leads to a sufficient condition for instability becomes apparent when one forms the characteristic equation for the linearized variational system associated with Eqs. (16) and (17), namely

$$\lambda^2 = -f_1 \cos\theta^* + f_2 \sin\theta^* - 2f_3 \cos 2\theta^* + 2f_4 \sin 2\theta^* \quad (25)$$

Clearly, this equation possesses a real, positive root whenever the right-hand member is positive. Hence $\theta = \theta^*$ is an *unstable* solution of Eq. (12) whenever θ^* satisfies Eq. (14) and

$$f_1 \cos\theta^* - f_2 \sin\theta^* + 2f_3 \cos 2\theta^* - 2f_4 \sin 2\theta^* < 0 \quad (26)$$

The significance and potential utility of these results can be clarified by reference to an illustrative example.

Letting L_1, L_2, and L_3 be principal axes of inertia of C for P, suppose that the axis of rotation passes through a point of L_1 and is parallel to L_3. The intersection of L_1 with the axis of rotation is then the point S, and, once $\mathbf{b}_x$ has been selected, the unit vectors $\mathbf{c}_1$, $\mathbf{c}_2$, and $\mathbf{c}_3$ must be drawn as shown in Fig. 4, that is, parallel to the principal axes of C for P, so that [see Eq. (2)]

$$I_{12} = I_{23} = I_{31} = 0 \quad (27)$$

As regards the placement of the axis of rotation in B, suppose that it passes through Q and is perpendicular to $\mathbf{a}_3$. Point T then coincides with S, the distance c is equal to zero,

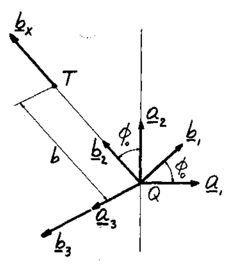

Fig. 5 Placement of axis of rotation in B.

and $\mathbf{b}_1$, $\mathbf{b}_2$, and $\mathbf{b}_3$ appear as shown in Fig. 5. Consequently

$$\mathbf{b}_x = \mathbf{b}_2 \tag{28}$$

and, in view of Eq. (1)

$$\mathbf{b}_y = \mathbf{b}_1 \tag{29}$$

so that, from Eqs. (3) and (4)

$$x_1 = 0, \qquad x_2 = 1, \qquad x_3 = 0 \tag{30}$$

and

$$y_1 = 1, \qquad y_2 = 0 \tag{31}$$

The coefficients $f_1, \ldots, f_4$, obtained by comparing Eq. (11) with Eq. (12), keeping in mind that $\phi = \phi_0$ and $\omega = 0$, and using Eqs. (27, 30, and 31), are now seen to be given by

$$f_1 = [3mbl/2(ml^2 + I_{33})]\Omega^2 \sin 2\phi_0 \tag{32}$$

$$f_2 = f_4 = 0 \tag{33}$$

$$f_3 = [(ml^2 - I_{11} + I_{22})/2(ml^2 + I_{33})]\Omega^2(1 + 3 \sin^2\phi_0) \tag{34}$$

Substitution from Eqs. (33) into Eq. (14) thus leads to the equilibrium condition in the form

$$f_1 \sin\theta^* + f_3 \sin 2\theta^* = 0 \tag{35}$$

and the stability condition (24) becomes

$$f_1 \cos\theta^* + 2f_3 \cos 2\theta^* > 0 \tag{36}$$

Eq. (35) possesses the three solutions

$$\theta^* = 0 \tag{37}$$

$$\theta^* = \pi \tag{38}$$

and

$$\theta^* = \cos^{-1}(-f_1/2f_3) \tag{39}$$

and the associated stability conditions are

$$f_1 + 2f_3 > 0 \tag{40}$$

$$-f_1 + 2f_3 > 0 \tag{41}$$

and

$$(f_1^2 - 4f_3^2)/2f_3 > 0 \tag{42}$$

The first two of these, together with Eqs. (32) and (34), show that one can choose parameter values either such that the two solutions $\theta = 0$ and $\theta = \pi$ are simultaneously stable, or such that they are simultaneously unstable, or such that one is stable while the other is unstable. As for the third equilibrium solution, it follows from Eq. (39) that this solution exists if and only if

$$f_1^2 - 4f_3^2 \leq 0 \tag{43}$$

so that the condition (42) is, in fact, equivalent to

$$f_3 < 0 \tag{44}$$

or, in view of Eq. (34)

$$ml^2 - I_{11} + I_{22} < 0 \tag{45}$$

In other words, the stability of this solution is independent of ϕ_0 and b. Potentially, this fact has practical significance, because by assigning suitable values to ϕ_0 and b, one can cause θ^* as given by Eq. (39) to take on any desired value, such as one of interest in establishing an attitude reference; and by choosing the inertia properties of C so as to satisfy the condition (45), one can then insure the stability of the solution in question.

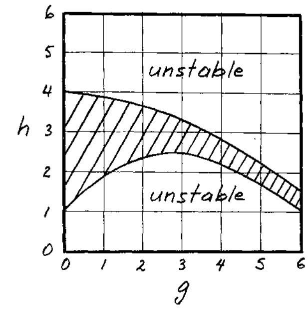

Fig. 6 Instability chart.

Rotating Satellites

If ω [see Eq. (10)] differs from zero, Eq. (13) is satisfied for all t when θ has a constant value θ^* such that

$$F_1(\theta^*) = F_2(\theta^*) = F_3(\theta^*) = 0 \tag{46}$$

and, if a new dependent variable, q, is introduced as

$$q = \theta - \theta^* \tag{47}$$

then Eq. (13) leads to

$$\ddot{q} + F_1(\theta^* + q) \sin 2(\phi_0 + \omega t) + F_2(\theta^* + q) \cos 2(\phi_0 + \omega t) + F_3(\theta^* + q) = 0 \tag{48}$$

which, in view of Eqs. (46), has the solution $q = 0$, the stability or instability of which implies the stability or instability of the solution $\theta = \theta^*$ of Eq. (13). Furthermore, expanding $F_1(\theta^* + q)$, $F_2(\theta^* + q)$, and $F_3(\theta^* + q)$ in Taylor series about θ^*, and retaining only terms linear in q, one obtains

$$\ddot{q} + [F_1' \sin 2(\phi_0 + \omega t) + F_2' \cos 2(\phi_0 + \omega t) + F_3']q \doteq 0 \tag{49}$$

where F_i' denotes the derivative of F_i with respect to θ, evaluated at $\theta = \theta^*$, and this equation can be brought into the standard form of Mathieu's equation by defining x, g, and h as

$$x = \phi_0 + \omega t - \pi/2 - \tan^{-1}(F_1'/F_2') \tag{50}$$

$$g = (1/2\omega^2)[(F_1')^2 + (F_2')^2]^{1/2} \tag{51}$$

and

$$h = F_3'/\omega^2 \tag{52}$$

That is, Eq. (49) can then be replaced with

$$(d^2q/dx^2) + (h - 2g \cos 2x)q = 0 \tag{53}$$

The relationship between the stability of the solution $q = 0$ of this equation, on the one hand, and the parameters g and h, on the other hand, has been discussed extensively.[4] In particular, it is known that this solution is unstable when g and h have values such that the point (g, h) falls into an unshaded region of Fig. 6, which consists of an enlarged portion of Fig. 8(A) of Ref. 4.

The fact that both g and h depend on ω points the way toward a practical application of the system under consideration, for it means that instabilities associated with changes in ω can be exploited for the purpose of constructing a simple device suitable for the detection of variations in the rotation rate of the satellite B. This can be accomplished by choosing the system parameters in such a way that the point in Fig. 6 associated with a desired value of ω lies in a relatively narrow portion of the shaded region, as is the case, for example, when $g = 3.5$ and $h = 2.5$, but that the points corresponding to values of ω slightly larger or smaller than the desired value lie in the unshaded portion. Small changes in ω will then lead to instability, and the concomitantly growing oscillations will reveal the fact that a change in ω has occurred. For instance, suppose that one wished to detect departures in ω/Ω from unity, with C mounted in B as in the previous

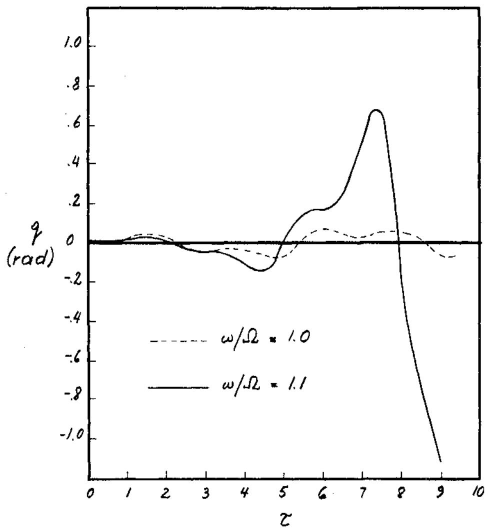

Fig. 7 Pendulum behavior.

example. Then Eqs. (27, 29, and 31) apply once again; the functions F_1, F_2, and F_3, obtained by comparing Eqs. (11) and (13) are given by

$$F_1(\theta) = [3mbl/2(ml^2 + I_{33})]\Omega^2 \sin\theta \quad (54)$$

$$F_2(\theta) = -[3(ml^2 - I_{11} + I_{22})/4(ml^2 + I_{33})]\Omega^2 \sin 2\theta \quad (55)$$

$$F_3(\theta) = [(ml^2 - I_{11} + I_{22})/4(ml^2 + I_{33})] \times (5\Omega^2 + 4\Omega\omega + 2\omega^2) \sin 2\theta \quad (56)$$

and, in accordance with Eqs. (46), the equilibrium values of θ are

$$\theta^* = 0 \quad (57)$$

and

$$\theta^* = \pi \quad (58)$$

Hence, arbitrarily selecting the second of these, one finds that

$$F_1' = -[3mbl/2(ml^2 + I_{33})]\Omega^2 \quad (59)$$

$$F_2' = -[3(ml^2 - I_{11} + I_{22})/2(ml^2 + I_{33})]\Omega^2 \quad (60)$$

$$F_3' = [(ml^2 - I_{11} + I_{22})/2(ml^2 + I_{33})] \times (5\Omega^2 + 4\Omega\omega + 2\omega^2) \quad (61)$$

and it follows from Eqs. (51) and (52) that

$$g = \{3[(mbl)^2 + (ml^2 - I_{11} + I_{22})^2]^{1/2}/4(ml^2 + I_{33})\} \times (\omega/\Omega)^{-2} \quad (62)$$

and

$$h = [(ml^2 - I_{11} + I_{22})/2(ml^2 + I_{33})] \times [5(\omega/\Omega)^{-2} + 4(\omega/\Omega)^{-1} + 2] \quad (63)$$

Consequently, if C is taken to be a thin, uniform circular disc of radius R, mounted in such a way that the axis of symmetry is parallel to the axis of rotation, so that (see Fig. 4)

$$I_{11} = I_{22}, \qquad I_{33} = mR^2/2 \quad (64)$$

then

$$g = \{3[1 + (b/l)^2]^{1/2}/2[2 + (R/l)^2]\}(\omega/\Omega)^{-2} \quad (65)$$

$$h = \{1/[2 + (R/l)^2]\}[5(\omega/\Omega)^{-2} + 4(\omega/\Omega)^{-1} + 2] \quad (66)$$

and the values of R/l and b/l which make $g = 3.5$ and $h = 2.5$ when $\omega/\Omega = 1$ are

$$R/l = (12/5)^{1/2}, \qquad b/l = [(154/15)^2 - 1]^{1/2} \quad (67)$$

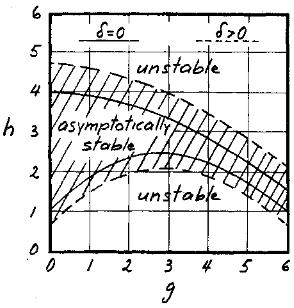

Fig. 8 Instability chart.

Thus, for $\omega/\Omega \neq 1$, substitution from Eqs. (67) into Eqs. (65) and (66) gives

$$g = (7/2)(\omega/\Omega)^{-2} \quad (68)$$

$$h = (5/22)[5(\omega/\Omega)^{-2} + 4(\omega/\Omega)^{-1} + 2] \quad (69)$$

and the effect on stability of departures of ω/Ω from the nominal value of unity can now be assessed by reference to Fig. 6. For example, with $\omega/\Omega = 1.1$, Eqs. (68) and (69) yield

$$g = 2.89, \qquad h = 2.22$$

and Fig. 6 indicates that the solution $\theta = \pi$ is unstable under these circumstances. The physical significance of this instability comes to light when one solves Eq. (48) numerically and then plots q vs τ, where $\tau = \Omega t$, both for $\omega/\Omega = 1$ and $\omega/\Omega = 1.1$, using the initial conditions $q(0) = 0.01$ rad and $\dot{q}(0) = 0$, these representing a situation in which the pendulum is initially at rest relative to the satellite, but is slightly displaced from its equilibrium position. Such plots are shown in Fig. 7, and the value of ω/Ω is seen to have a pronounced effect on the behavior of the pendulum: For $\omega/\Omega = 1$, q remains smaller than ten times its initial value throughout the time interval under consideration, whereas, if $\omega/\Omega = 1.1$, q acquires values larger than one hundred times the initial value. Similar results are obtained when $\omega/\Omega = 0.9$, in which case Eqs. (68) and (69) lead to

$$g = 4.32, \qquad h = 2.87$$

so that, in accordance with Fig. 6, the solution $\theta = \pi$ is again unstable. A thin, uniform circular disc, mounted as required by Eqs. (67), can thus serve as a device for the detection of both positive and negative ten per cent departures of ω/Ω from the nominal value of unity.

5. Friction Effects

The results obtained so far are applicable only when C is completely free to rotate relative to B. As a small amount of frictional resistance cannot be avoided in practice, it is worthwhile to attempt to assess the possible effects of such friction by postulating a resistance torque whose magnitude is proportional to the angular speed of C relative to B. A term of the form $\delta\dot{\theta}$, where δ is a positive constant, must then be added to the left-hand member of each of Eqs. (12) and (13). The addition of this term does not alter the equilibrium conditions stated in Eqs. (14) and (46), but it requires the replacement of Eq. (25) with

$$\lambda^2 + \delta\lambda = -f_1 \cos\theta^* + f_2 \sin\theta^* - 2f_3 \cos 2\theta^* + 2f_4 \sin 2\theta^* \quad (70)$$

and the replacement of Eq. (53) with

$$d^2q/dx^2 + \delta dq/dx + (h - 2g \cos 2x)q = 0 \quad (71)$$

Both roots of Eq. (70) have negative real parts if the condition (24) is satisfied, and at least one root of Eq. (70) has a positive real part when the inequality (26) is satisfied. For the case of the Earth-pointing satellite, the effect of the postulated friction is thus simply to convert marginal stability into asymptotic stability. As for the case of the rotating satellite, the relationship between the stability of the solution $q = 0$ of Eq. (71) and the parameters g, h, and δ has been shown by V. G. Kotowski[5] to be such that, for $\delta > 0$, Fig. 6 is to be replaced with Fig. 8, in which the unshaded region once again corresponds to instability, and the shaded region is now associated with asymptotic stability. Thus it may be surmised that friction effects will be at worst, innocuous, and at best, helpful.

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Suboptimal Control of Linear Systems Derived from Models of Lower Dimension

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The problem considered is the following: given a linear dynamic system of order n, find the best model of order m, $m < n$, with which to derive a suboptimal control for the given system. The optimization problem covered is the infinite time, linear, output regulator problem with quadratic cost. The system and model outputs are characterized as elements of an appropriate Hilbert space, and the model output is constrained to be a projection of the system output. In this manner, the optimal model initial condition is expressed as a linear function of the system initial condition, and an algebraic expression is found for the modeling error which is minimized numerically. An example is presented wherein the pitch plane dynamics of a flexible-bodied rocket vehicle are modeled.

I. Introduction

SIMPLIFYING mathematical models of dynamic systems has traditionally relied heavily on the experience and ingenuity of the analyst. More recently, there has been considerable activity in the area of developing general techniques for simplifying, or approximating linear models of dynamic systems. These efforts have principally addressed the problem of modeling a system of linear homogeneous differential equations with a linear homogeneous model of lower order; usually constant coefficient. Modeling has been achieved using both frequency and time domain techniques, and the approach has varied in sophistication from pole removal to projection in a function space.[1,2]

This paper is concerned with the following problem: given a linear dynamic system of order n, find the best model of order m, $m < n$, with which to derive a suboptimal control for the given system. The optimization problem covered is the infinite time, linear, output regulator problem with quadratic cost. A model is derived such that the optimal control policy for the model is the "best" suboptimal control policy for the actual system.

Simplified models are important in applications of optimal control theory. For example, the synthesis of a stability augmentation system for a helicopter using the formalism of Murphy and Narenda involves solving successively several optimization problems with concurrent simulation of the system response to a variety of initial conditions.[3] If the dimension of the state vector is high, the computation time required by this procedure becomes excessive. The designer is forced to arbitrarily reduce the order of the model to an order compatible with the computer time allotted to the design. In other problems control-system simplicity is essential and a dynamic "observer" to reconstruct the components of the state vector not measured would be too complex. In this event a control policy that requires only a linear algebraic operation on the system outputs would have obvious advantages.

The modeling technique presented here provides a means of deriving a simple model and a suboptimal control policy that operates directly on the measured state variables. The approach to the modeling problem taken here is to characterize the system and model outputs as elements of a Hilbert space. The equations used in deriving the model are then obtained by constraining the model solutions to be projections of the

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